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Block LU factors of generalized companion matrix pencils

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Abstract

We present formulas for computations involving companion matrix pencils as may arise in considering polynomial eigenvalue problems. In particular, we provide explicit companion matrix pencils for matrix polynomials expressed in a variety of polynomial bases including monomial, orthogonal, Newton, Lagrange, and Bernstein/Bézier bases. Additionally, we give a pair of explicit **LU** factors associated with each pencil and a prescription for block pivoting when required.

Keywords: Matrix polynomials; Polynomial eigenvalues; Block **LU** factoring

1. Introduction

A matrix polynomial is usually defined as a matrix-valued function of the form

$$\mathbf{P}(x) = \sum_{j=0} x^k \mathbf{A}_j = \mathbf{A}_0 + x\,\mathbf{A}_1 + \cdots x^n\,\mathbf{A}_n, \tag{1}$$

where x is a scalar argument and the coefficients $\mathbf{A}_j$, $(j = 0\colon n)$ are $s \times s$ matrices (see [13,15]). A common problem related to a matrix polynomial $\mathbf{P}(x)$ is to find the values of $x \in \mathbb{C}$ such that $\mathbf{P}(x)$ is a singular matrix; these values are the *polynomial eigenvalues* (or *latent roots*) of the matrix polynomial $\mathbf{P}(x)$ and the problem of determining such values is referred to as the polynomial eigenvalue problem (sometimes nonlinear eigenvalue problem [10]).

Typically, a polynomial eigenvalue problem is solved by solving an associated generalized eigenvalue problem. That is, a pair of matrices $(\mathbf{C}_0, \mathbf{C}_1)$ is found such that the linear matrix polynomial $x\,\mathbf{C}_1 - \mathbf{C}_0$ is singular at the same values of $x \in \mathbb{C}$ at which $\mathbf{P}(x)$ is singular. The pair $(\mathbf{C}_0, \mathbf{C}_1)$ is called a *generalized companion matrix pencil* corresponding to the matrix polynomial $\mathbf{P}(x)$. Under certain suitable conditions, such a matrix pencil can be called a linearization of the matrix polynomial [13].

The expression in (1) is the most familiar description of matrix polynomials written in the monomial basis $\{1, x, x^2, \ldots, x^n\}$. However, as with the scalar polynomial case, matrix polynomials can be written relative to

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polynomial bases other than the monomial basis. Bases other than the monomial basis find many applications. For problems in computer-aided geometric design, the Bernstein–Bézier basis and the Lagrange basis are most useful; for special problems, e.g. where symmetry in boundary conditions in PDE control series expansions, orthogonal bases such as the Legendre polynomials are often the most natural; finally, in approximation theory, Chebyshev polynomials have a special place due to their minimum-norm property [21]. In this paper, we provide some tools for rapid and accurate computation with matrix polynomials expressed in any of these.

Specifically, we describe explicit companion matrix pencils $x\mathbf{C}_1 - \mathbf{C}_0$ associated with matrix polynomials represented in the polynomial bases just described. We also provide block **LU** factors of each matrix $x\,\mathbf{C}_1 - \mathbf{C}_0$ in addition to the generalized companion matrix pencils in each polynomial basis considered. The **LU** factorings presented are inexpensive formulas that are simple to implement. One motivation for the derivation of these **LU** factors is for adapting Rayleigh quotient iteration for determining polynomial eigenvalues [19]. That is, given a matrix polynomial $\mathbf{P}(x)$ and an associated generalized companion matrix pencil, each step of Rayleigh quotient iteration proceeds starting with approximate right and left eigenvectors $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ respectively of the pair $(\mathbf{C}_0, \mathbf{C}_1)$. We can generate an approximate eigenvalue $\tilde{x}$ by the formula

$$\tilde{x} := \frac{\tilde{\mathbf{y}}^H \mathbf{C}_0 \tilde{\mathbf{x}}}{\tilde{\mathbf{y}}^H \mathbf{C}_1 \tilde{\mathbf{x}}}. \tag{2}$$

In the event that $\mathbf{C}_1$ is the identity matrix, the expression in (2) is the usual Rayleigh quotient. To generate updated approximate eigenvectors, linear systems of the form

$$(\tilde{x}\,\mathbf{C}_1 - \mathbf{C}_0)\,\mathbf{z} = \mathbf{C}_1\tilde{\mathbf{x}}, \tag{3a}$$

$$\mathbf{w}^H\,(\tilde{x}\,\mathbf{C}_1 - \mathbf{C}_0) = \tilde{\mathbf{y}}^H\mathbf{C}_1 \tag{3b}$$

need to be solved within each iteration (e.g., see [3]). Thus, an **LU** factoring of $\tilde{x}\,\mathbf{C}_1 - \mathbf{C}_0$ can be used to solve efficiently systems like (3) that would arise in Rayleigh quotient iteration or other similar algorithms based on the inverse power method. We do, of course, need to take notice of stability considerations; the closer $\tilde{x}$ is to an actual polynomial eigenvalue, the closer $\tilde{x}\,\mathbf{C}_1 - \mathbf{C}_0$ is to a singular matrix.

There are other reasons one might desire a structured **LU** factoring of these special matrices, but they usually boil down to the desire for fast and accurate solution of these structured systems. For example, to estimate the norm of the resolvent $\|(x\mathbf{C}_1 - \mathbf{C}_0)^{-1}\|$, these factors are useful, but only because an estimate requires solving systems like (3).

All of the generalized companion matrix pencils used in the present work are used in [8] except for the pencil (38) associated with the Lagrange basis [9] and the pencil of the form (30) and (31) associated with the Bernstein basis [7,18,24]. The pencil $x\,\mathbf{C}_1 - \mathbf{C}_0$ is explicitly inverted in [23]. What is new in this paper are the specific factorings.

We begin in Section 2 by considering the family of degree-graded bases. We present an explicit representation of a companion matrix pencil in a typical degree-graded polynomial basis. We show the block **LU** factors of this companion matrix pencil. In addition, we provide costs for construction and solving a system using these **LU** factors. We focus on the stability of the **LU** factors of this companion matrix pencil, specifying a special-purpose block partial-pivoting method to stabilize them when necessary. In Section 3, we consider some of the most famous degree-graded bases. These bases include any bases of orthogonal polynomials, the standard monomial (or power) basis and the Newton basis (with the Pochhammer basis as a special case). Sections 4 and 5 are about the Bernstein basis and the Lagrange basis, respectively and give the same information about these two bases as Section 2 does about the degree-graded bases. These bases are not degree-graded. Of course, the Bernstein basis is similar to degree-graded bases.

Throughout the present work, boldface letters are used to denote vectors and matrices. We employ the colon notation to indicate ranges of integer variables (as in MATLAB). The superscript $()^H$ denotes the Hermitian (complex-conjugate) transpose of a matrix or vector while the superscript $()^{\mathrm{T}}$ denotes the transpose *without* complex conjugation. The notation $\otimes$ denotes the Kronecker (alternatively, direct or tensor) product of matrices [17, p. 242]. To facilitate the discussion of block pivoting, the permutation matrix that exchanges rows j and k in an $n \times n$ matrix is

$$\mathbf{E}_{[j\leftrightarrow k]} := \mathbf{I}_n - \mathbf{e}_j\mathbf{e}_j^{\mathrm{T}} - \mathbf{e}_k\mathbf{e}_k^{\mathrm{T}} + \mathbf{e}_j\mathbf{e}_k^{\mathrm{T}} + \mathbf{e}_k\mathbf{e}_j^{\mathrm{T}}, \quad j, k \in \{1, \ldots, n\},\ j \neq k, \tag{4}$$

where $\mathbf{e}_k \in \mathbb{C}^n$ is the k-th coordinate column vector.

We use the usual Landau "big-Oh" notation $\mathcal{O}$ to discuss the asymptotic complexity of various formulas. The costs associated with our formulas typically depend simultaneously on n (the degree of the matrix polynomial) and s (the size of the matrix blocks). $\mathcal{O}(n^2 s + s^3)$ means that the cost is $\mathcal{O}(n^2)$ for fixed s as $n \to \infty$ (with a constant factor that includes s) and simultaneously the cost is $\mathcal{O}(s^3)$ for fixed n and $s \to \infty$. This shorthand should not cause confusion, as simultaneous limits $n \to \infty$ and $s \to \infty$ are not considered (e.g., see [20]).

2. Degree-graded polynomial bases

In this section, we consider a family of real polynomials $\{\phi_j(x)\}_{j=0}^{\infty}$ with $\phi_j(x)$ of degree j which satisfy a three-term recurrence relation. In fact we have:

$$x\phi_j(x) = \alpha_j \phi_{j+1}(x) + \beta_j \phi_j(x) + \gamma_j \phi_{j-1}(x), \quad j = 0, 1, 2, \ldots, \tag{5}$$

where the $\alpha_j,\ \beta_j,\ \gamma_j$ are real, $\phi_{-1}(x) = 0$, $\phi_0(x) = 1$, and, if k_j is the leading coefficient of $\phi_j(x)$,

$$0 \neq \alpha_j = \frac{k_j}{k_{j+1}}, \quad j = 0, 1, 2, \ldots. \tag{6}$$

An $s \times s$ matrix polynomial $\mathbf{P}(x)$ of degree n can now be written in terms of a set of degree-graded polynomials

$$\mathbf{P}(x) = \mathbf{A}_n \phi_n(x) + \mathbf{A}_{n-1}\phi_{n-1}(x) + \cdots + \mathbf{A}_1 \phi_1(x) + \mathbf{A}_0 \phi_0(x). \tag{7}$$

2.1. Companion matrix pencil

Consider the matrix polynomial $\mathbf{P}(x)$ of the form (7). There exists a pair of block matrices $\mathbf{C}_0, \mathbf{C}_1 \in \mathbb{C}^{ns \times ns}$ such that the generalized eigenvalues of the pencil $(\mathbf{C}_0, \mathbf{C}_1)$ are the same as the polynomial eigenvalues of $\mathbf{P}(x)$ (e.g., see [13]). Now, using (5) and (6) and through a little computation, we can construct the companion matrix pencils for degree-graded polynomials. For convenience, here we set $n = 5$, but the generalizations for all positive n are obvious. The companion matrix pencils are as follows in this case:

$$\mathbf{C}_0 = \begin{bmatrix} \beta_0 \mathbf{I}_s & \gamma_1 \mathbf{I}_s & & & -k_4 \mathbf{A}_0 \\ \alpha_0 \mathbf{I}_s & \beta_1 \mathbf{I}_s & \gamma_2 \mathbf{I}_s & & -k_4 \mathbf{A}_1 \\ & \alpha_1 \mathbf{I}_s & \beta_2 \mathbf{I}_s & \gamma_3 \mathbf{I}_s & -k_4 \mathbf{A}_2 \\ & & \alpha_2 \mathbf{I}_s & \beta_3 \mathbf{I}_s & -k_4 \mathbf{A}_3 + k_5 \gamma_4 \mathbf{A}_5 \\ & & & \alpha_3 \mathbf{I}_s & -k_4 \mathbf{A}_4 + k_5 \beta_4 \mathbf{A}_5 \end{bmatrix}, \tag{8}$$

$$\mathbf{C}_1 = \begin{bmatrix} \mathbf{I}_s & & & & \\ & \mathbf{I}_s & & & \\ & & \mathbf{I}_s & & \\ & & & \mathbf{I}_s & \\ & & & & k_5 \mathbf{A}_5 \end{bmatrix}. \tag{9}$$

In fact, we get the same thing as "comrade" matrices in [5,8].

2.2. Explicit LU factors and complexity

Let $\mathbf{P}(x)$ be expressed relative to a basis of degree-graded polynomials as in (7). Using (8) and (9), the matrix $x\mathbf{C}_1 - \mathbf{C}_0$ can be written as

$$(x\mathbf{C}_1 - \mathbf{C}_0) = \begin{bmatrix} (x-\beta_0)\mathbf{I}_s & -\gamma_1 \mathbf{I}_s & & & k_4 \mathbf{A}_0 \\ -\alpha_0 \mathbf{I}_s & (x-\beta_1)\mathbf{I}_s & -\gamma_2 \mathbf{I}_s & & k_4 \mathbf{A}_1 \\ & -\alpha_1 \mathbf{I}_s & (x-\beta_2)\mathbf{I}_s & -\gamma_3 \mathbf{I}_s & k_4 \mathbf{A}_2 \\ & & -\alpha_2 \mathbf{I}_s & (x-\beta_3)\mathbf{I}_s & k_4 \mathbf{A}_3 - k_5 \gamma_4 \mathbf{A}_5 \\ & & & -\alpha_3 \mathbf{I}_s & k_4 \mathbf{A}_4 + k_5 (x-\beta_4) \mathbf{A}_5 \end{bmatrix}. \tag{10}$$

If x is not a root of any of the polynomials $\phi_j(x)$ $(j = 1:n-1)$, a little computation using (5) shows that (10) admits a block $\mathbf{LU}$ factoring without pivoting; that is, $x\mathbf{C}_1 - \mathbf{C}_0 = \mathbf{LU}$, where

$$\mathbf{L} = \begin{bmatrix} \mathbf{I}_s & & & & \\ -\frac{\phi_0(x)}{\phi_1(x)}\mathbf{I}_s & \mathbf{I}_s & & & \\ & -\frac{\phi_1(x)}{\phi_2(x)}\mathbf{I}_s & \mathbf{I}_s & & \\ & & -\frac{\phi_2(x)}{\phi_3(x)}\mathbf{I}_s & \mathbf{I}_s & \\ & & & -\frac{\phi_3(x)}{\phi_4(x)}\mathbf{I}_s & \mathbf{I}_s \end{bmatrix}, \tag{11a}$$

$$\mathbf{U} = \begin{bmatrix} \alpha_0\frac{\phi_1(x)}{\phi_0(x)}\mathbf{I}_s & -\gamma_1\mathbf{I}_s & & & \mathbf{U}_{1,5} \\ & \alpha_1\frac{\phi_2(x)}{\phi_1(x)}\mathbf{I}_s & -\gamma_2\mathbf{I}_s & & \mathbf{U}_{2,5} \\ & & \alpha_2\frac{\phi_3(x)}{\phi_2(x)}\mathbf{I}_s & -\gamma_3\mathbf{I}_s & \mathbf{U}_{3,5} \\ & & & \alpha_3\frac{\phi_4(x)}{\phi_3(x)}\mathbf{I}_s & \mathbf{U}_{4,5} \\ & & & & \mathbf{U}_{5,5} \end{bmatrix}, \tag{11b}$$

where for general n, the block entries $\mathbf{U}_{j,n}$ are given as follows

$$\mathbf{U}_{1,n} = k_{n-1}\mathbf{A}_0, \tag{11c}$$

$$\mathbf{U}_{j,n} = k_{n-1}\mathbf{A}_{j-1} + \frac{\phi_{j-2}(x)}{\phi_{j-1}(x)}\mathbf{U}_{j-1,n}, \quad j = 2:n-2, \tag{11d}$$

$$\mathbf{U}_{n-1,n} = k_{n-1}\mathbf{A}_{n-2} + \frac{\phi_{n-3}(x)}{\phi_{n-2}(x)}\mathbf{U}_{n-2,n} - k_n\gamma_{n-1}\mathbf{A}_n, \tag{11e}$$

$$\mathbf{U}_{n,n} = \frac{\phi_0(x)}{\phi_{n-1}(x)(\alpha_0\cdots\alpha_{n-2})}\mathbf{P}(x). \tag{11f}$$

The degree-graded polynomials $\phi_j(x)$ can be evaluated at x using (5) in at most $\mathcal{O}(n)$ operations. Assuming x is distinct from the roots of all the polynomials $\phi_j(x)$ $(j = 1:n-1)$, the matrices $\mathbf{L}$ and $\mathbf{U}$ can be built in $\mathcal{O}(ns^2)$ work with the rightmost block column of $\mathbf{U}$ requiring the most work. Similarly, the factoring in (11) shows that the system (3) can be solved with matrices of the form (10) as the coefficients in $\mathcal{O}(ns^2 + s^3)$ operations whenever $\mathbf{P}(x)$ is nonsingular.

The $\mathbf{LU}$ factors derived in this section are subject to certain limitations. In particular the formulas for the factors $\mathbf{L}$ and $\mathbf{U}$ of the matrix $x\mathbf{C}_1 - \mathbf{C}_0$ require block pivoting when x is a zero of one of $\phi_j(x)$ $(j = 1:n-1)$. We present the revised factorings for the degree-graded companion matrix pencils below along with the block permutation matrices needed in their construction.

2.3. *Block pivoting strategies for stabilization of* LU *factors*

Consider the generalized companion matrix pencil of the form (10) for a matrix polynomial $\mathbf{P}(x)$ written using a degree-graded polynomial basis. If x is a root of any of the polynomials $\phi_j(x)$ $(j = 1:n)$, (or even close to one of them) it is preferable to premultiply (10) by the block permutation matrix

$$\mathbf{S} := \left(\mathbf{E}_{[n\leftrightarrow n+1]}\mathbf{E}_{[n-1\leftrightarrow n]}\cdots\mathbf{E}_{[1\leftrightarrow 2]}\right)\otimes\mathbf{I}_s = \begin{bmatrix} & \mathbf{I}_s & & \\ & & \ddots & \\ & & & \mathbf{I}_s \\ \mathbf{I}_s & & & \end{bmatrix}. \tag{12}$$

This produces a factoring that is continuous as x approaches a root of $\phi_j(x)$, and can therefore be expected to be more numerically stable. With $\mathbf{S}$ defined as (12), for $n = 5$, with the clear generalizations for all positive n, we have

$$\mathbf{S}(x\mathbf{C}_1 - \mathbf{C}_0) = \begin{bmatrix} -\alpha_0\mathbf{I}_s & (x-\beta_1)\mathbf{I}_s & -\gamma_2\mathbf{I}_s & & k_4\mathbf{A}_1 \\ & -\alpha_1\mathbf{I}_s & (x-\beta_2)\mathbf{I}_s & -\gamma_3\mathbf{I}_s & k_4\mathbf{A}_2 \\ & & -\alpha_2\mathbf{I}_s & (x-\beta_3)\mathbf{I}_s & k_4\mathbf{A}_3 - k_5\gamma_4\mathbf{A}_5 \\ & & & -\alpha_3\mathbf{I}_s & k_4\mathbf{A}_4 + k_5(x-\beta_4)\mathbf{A}_5 \\ (x-\beta_0)\mathbf{I}_s & -\gamma_1\mathbf{I}_s & & & k_4\mathbf{A}_0 \end{bmatrix}, \tag{13}$$

and with a little computation using (5), it turns out that an **LU** factoring of a matrix of the form (13) and of degree n is as follows:

$$\mathbf{L} = \begin{bmatrix} \mathbf{I}_s & & & & \\ & \mathbf{I}_s & & & \\ & & \mathbf{I}_s & & \\ & & & \mathbf{I}_s & \\ -\frac{\phi_1(x)}{\phi_0(x)}\mathbf{I}_s & -\frac{\phi_2(x)}{\phi_0(x)}\mathbf{I}_s & -\frac{\phi_3(x)}{\phi_0(x)}\mathbf{I}_s & -\frac{\phi_4(x)}{\phi_0(x)}\mathbf{I}_s & \mathbf{I}_s \end{bmatrix}, \tag{14a}$$

$$\mathbf{U} = \begin{bmatrix} -\alpha_0\mathbf{I}_s & (x-\beta_1)\mathbf{I}_s & -\gamma_2\mathbf{I}_s & & k_4\mathbf{A}_1 \\ & -\alpha_1\mathbf{I}_s & (x-\beta_2)\mathbf{I}_s & -\gamma_3\mathbf{I}_s & k_4\mathbf{A}_2 \\ & & -\alpha_2\mathbf{I}_s & (x-\beta_3)\mathbf{I}_s & k_4\mathbf{A}_3 - k_5\gamma_4\mathbf{A}_5 \\ & & & -\alpha_3\mathbf{I}_s & k_4\mathbf{A}_4 + k_5(x-\beta_4)\mathbf{A}_5 \\ & & & & \frac{1}{\alpha_0\alpha_1\alpha_2\alpha_3}\mathbf{P}(x) \end{bmatrix}. \tag{14b}$$

For general n, the block entries $\mathbf{U}_{j,n}$ are given as follows

$$\mathbf{U}_{j,n} = k_{n-1}\mathbf{A}_j, \quad j = 1:n-3, \tag{14c}$$

$$\mathbf{U}_{n-2,n} = k_{n-1}\mathbf{A}_{n-2} - k_n\gamma_{n-1}\mathbf{A}_n, \tag{14d}$$

$$\mathbf{U}_{n-1,n} = k_{n-1}\mathbf{A}_{n-1} + k_n(x-\beta_{n-1})\mathbf{A}_n, \tag{14e}$$

$$\mathbf{U}_{n,n} = \frac{1}{\alpha_0\cdots\alpha_{n-2}}\mathbf{P}(x). \tag{14f}$$

The cost of building (14) with pivoting is the same as the cost of building (11) without pivoting. However, it should be noted that the factors in (14) are not appropriate when $|\phi_j(x)| \gg 1$ ($j = 1:n-1$). In particular, in the 1-norm, the condition number of **L** in (14a) is

$$\kappa_1(\mathbf{L}) = (\max(1+|\phi_1(x)|, \ldots, 1+|\phi_{n-1}(x)|))^2. \tag{15}$$

3. Special degree-graded bases

In this section, we consider some of the most famous bases that fall into the category of degree-graded polynomials.

3.1. Orthogonal bases

If all α_j and γ_j in (6) are positive, then $\phi_j(x)$ form an orthogonal basis (see e.g. [5]). Orthogonal polynomials such as Legendre or Chebyshev polynomials have well-established significance in mathematical physics and numerical analysis (see, e.g. [12]). The elements of an orthogonal polynomial basis are polynomials $\phi_j(x)$ that are orthogonal over a particular interval $[a, b]$ with respect to some associated weight function [1, Chapter 22]. All the roots of $\phi_j(x)$ lie between a and b.

The **LU** factoring of a companion matrix pencil in an orthogonal basis is given in (11) and possibly (14). If x is very far from the interval $[a, b]$ containing the roots of the basis polynomials, from (15) it can be seen that the condition number in the 1-norm of **L** is exponential in n.

3.2. Monomial basis

This is the most commonly used basis among the others. If we let $\alpha_j = 1$, $\beta_j = 0$ and $\gamma_j = 0$ in (6), then $\phi_j(x) = x^j$ form the monomial basis. In that case (10) for $n = 5$ becomes:

$$(x\mathbf{C}_1 - \mathbf{C}_0) = \begin{bmatrix} x\mathbf{I}_s & & & & \mathbf{A}_0 \\ -\mathbf{I}_s & x\mathbf{I}_s & & & \mathbf{A}_1 \\ & -\mathbf{I}_s & x\mathbf{I}_s & & \mathbf{A}_2 \\ & & -\mathbf{I}_s & x\mathbf{I}_s & \mathbf{A}_3 \\ & & & -\mathbf{I}_s & \mathbf{A}_4 + x\mathbf{A}_5 \end{bmatrix}. \tag{16}$$

The above pencil is used by Matlab's `polyeig` function to compute the polynomial eigenvalues of a matrix polynomial.

Using (11), we can write the **LU** factors of (16) as follows:

$$\mathbf{L} = \begin{bmatrix} \mathbf{I}_s & & & & \\ -\frac{1}{x}\mathbf{I}_s & \mathbf{I}_s & & & \\ & -\frac{1}{x}\mathbf{I}_s & \mathbf{I}_s & & \\ & & -\frac{1}{x}\mathbf{I}_s & \mathbf{I}_s & \\ & & & -\frac{1}{x}\mathbf{I}_s & \mathbf{I}_s \end{bmatrix}, \tag{17a}$$

$$\mathbf{U} = \begin{bmatrix} x\mathbf{I}_s & & & & \mathbf{U}_{1,5} \\ & x\mathbf{I}_s & & & \mathbf{U}_{2,5} \\ & & x\mathbf{I}_s & & \mathbf{U}_{3,5} \\ & & & x\mathbf{I}_s & \mathbf{U}_{4,5} \\ & & & & \mathbf{U}_{5,5} \end{bmatrix}, \tag{17b}$$

where for general n, the block entries $\mathbf{U}_{j,n}$ are given as follows

$$\mathbf{U}_{1,n} = \mathbf{A}_0, \tag{17c}$$

$$\mathbf{U}_{j,n} = \mathbf{A}_{j-1} + \frac{1}{x}\mathbf{U}_{j-1,n}, \quad j = 2{:}n-1, \tag{17d}$$

$$\mathbf{U}_{n,n} = \frac{1}{x^{n-1}}\mathbf{P}(x). \tag{17e}$$

When $x \to 0$, using the special block pivoting strategy described in Section 2.3 and (14), we have the following **LU** factors:

$$\mathbf{L} = \begin{bmatrix} \mathbf{I}_s & & & & \\ & \mathbf{I}_s & & & \\ & & \mathbf{I}_s & & \\ & & & \mathbf{I}_s & \\ -x\mathbf{I}_s & -x^2\mathbf{I}_s & -x^3\mathbf{I}_s & -x^4\mathbf{I}_s & \mathbf{I}_s \end{bmatrix}, \tag{18a}$$

$$\mathbf{U} = \begin{bmatrix} -\mathbf{I}_s & x\mathbf{I}_s & & & \mathbf{A}_1 \\ & -\mathbf{I}_s & x\mathbf{I}_s & & \mathbf{A}_2 \\ & & -\mathbf{I}_s & x\mathbf{I}_s & \mathbf{A}_3 \\ & & & -\mathbf{I}_s & \mathbf{A}_4 + x\mathbf{A}_5 \\ & & & & \mathbf{P}(x) \end{bmatrix}. \tag{18b}$$

For general n, the block entries $\mathbf{U}_{j,n}$ are given as follows

$$\mathbf{U}_{j,n} = \mathbf{A}_j, \quad j = 1{:}n-2, \tag{18c}$$

$$\mathbf{U}_{n-1,n} = \mathbf{A}_{n-1} + x\mathbf{A}_n, \tag{18d}$$

$$\mathbf{U}_{n,n} = \mathbf{P}(x). \tag{18e}$$

For $|x| \gg 1$ the factors in (18) are not appropriate. In particular, as a special case of (15), in the 1-norm, the condition number of $\mathbf{L}$ in (18a) is

$$\kappa_1(\mathbf{L}) = (\max(1+|x|, \ldots, 1+|x^{n-1}|))^2. \tag{19}$$

3.3. Newton basis

An $s \times s$ matrix polynomial specified by data $\{(x_j, \mathbf{P}_j)\}_{j=0}^{n}$ can be represented using the Newton basis. The Newton basis $\{N_j(x)\}_{j=0}^{n}$ is given by

$$N_j(x) = \prod_{i=0}^{j-1}(x - x_i), \quad j = 0{:}n, \tag{20}$$

where the empty product $N_0(x) \equiv 1$. In fact, the monomial basis is a special case of the Newton basis where all the sample points $x_j = 0$ ($j = 0{:}n$).

Given (20), the matrix polynomial $\mathbf{P}(x)$ expressed in the Newton basis is

$$\mathbf{P}(x) = \sum_{j=0}^{n} N_j(x)\mathbf{A}_j, \tag{21}$$

where the coefficients $\mathbf{A}_j$ are determined using divided differences or, equivalently, by solving the block matrix system of linear equations

$$\begin{bmatrix} \mathbf{I}_s & & & & \\ \mathbf{I}_s & N_1(x_1)\,\mathbf{I}_s & & & \\ \mathbf{I}_s & N_1(x_2)\,\mathbf{I}_s & N_2(x_2)\,\mathbf{I}_s & & \\ \vdots & \vdots & \vdots & \ddots & \\ \mathbf{I} & N_1(x_n)\,\mathbf{I}_s & N_2(x_n)\,\mathbf{I}_s & \cdots & N_n(x_n)\,\mathbf{I}_s \end{bmatrix} \begin{bmatrix} \mathbf{A}_0 \\ \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_n \end{bmatrix} = \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \vdots \\ \mathbf{P}_n \end{bmatrix}, \tag{22}$$

and the cost of this computation is $\mathcal{O}(n^2 s^2)$.

If we let $\alpha_j = 1$, $\beta_j = x_j$ and $\gamma_j = 0$ in (6), then we will get the Newton basis. If the matrix polynomial $\mathbf{P}(x)$ is specified by its Newton basis coefficients $\mathbf{A}_j$, using (10), an associated companion matrix pencil for $n = 5$ is

$$(x\mathbf{C}_1 - \mathbf{C}_0) = \begin{bmatrix} (x-x_0)\mathbf{I}_s & & & & \mathbf{A}_0 \\ -\mathbf{I}_s & (x-x_1)\mathbf{I}_s & & & \mathbf{A}_1 \\ & -\mathbf{I}_s & (x-x_2)\mathbf{I}_s & & \mathbf{A}_2 \\ & & -\mathbf{I}_s & (x-x_3)\mathbf{I}_s & \mathbf{A}_3 \\ & & & -\mathbf{I}_s & \mathbf{A}_4 + (x-x_4)\mathbf{A}_5 \end{bmatrix}. \tag{23}$$

Now from (11), the $\mathbf{LU}$ factoring of a matrix of the from (23) will be as follows:

$$\mathbf{L} = \begin{bmatrix} \mathbf{I}_s & & & & \\ -\frac{1}{x-x_0}\mathbf{I}_s & \mathbf{I}_s & & & \\ & -\frac{1}{x-x_1}\mathbf{I}_s & \mathbf{I}_s & & \\ & & -\frac{1}{x-x_2}\mathbf{I}_s & \mathbf{I}_s & \\ & & & -\frac{1}{x-x_3}\mathbf{I}_s & \mathbf{I}_s \end{bmatrix}, \tag{24a}$$

$$\mathbf{U} = \begin{bmatrix} (x-x_0)\mathbf{I}_s & & & & \mathbf{U}_{1,5} \\ & (x-x_1)\mathbf{I}_s & & & \mathbf{U}_{2,5} \\ & & (x-x_2)\mathbf{I}_s & & \mathbf{U}_{3,5} \\ & & & (x-x_3)\mathbf{I}_s & \mathbf{U}_{4,5} \\ & & & & \mathbf{U}_{5,5} \end{bmatrix}, \tag{24b}$$

where for general n, the block entries $\mathbf{U}_{j,n}$ are given as follows

$$\mathbf{U}_{1,n} = \mathbf{A}_0, \tag{24c}$$

$$\mathbf{U}_{j,n} = \mathbf{A}_{j-1} + \frac{1}{x - x_{j-2}}\mathbf{U}_{j-1,n}, \quad j = 2{:}n-1, \tag{24d}$$

$$\mathbf{U}_{n,n} = \frac{1}{(x-x_0)\cdots(x-x_{n-2})}\mathbf{P}(x). \tag{24e}$$

When $x \to x_j$ ($j = 0{:}n-2$), through the special block pivoting strategy described in Section 2.3, (14) becomes

$$\mathbf{L} = \begin{bmatrix} \mathbf{I}_s & & & & \\ & \mathbf{I}_s & & & \\ & & \mathbf{I}_s & & \\ & & & \mathbf{I}_s & \\ -(x-x_0)\mathbf{I}_s & -\prod_{i=0}^{1}(x-x_i)\mathbf{I}_s & -\prod_{i=0}^{2}(x-x_i)\mathbf{I}_s & -\prod_{i=0}^{3}(x-x_i)\mathbf{I}_s & \mathbf{I}_s \end{bmatrix}, \tag{25a}$$

$$\mathbf{U} = \begin{bmatrix} -\mathbf{I}_s & (x-x_1)\mathbf{I}_s & & & \mathbf{A}_1 \\ & -\mathbf{I}_s & (x-x_2)\mathbf{I}_s & & \mathbf{A}_2 \\ & & -\mathbf{I}_s & (x-x_3)\mathbf{I}_s & \mathbf{A}_3 \\ & & & -\mathbf{I}_s & \mathbf{A}_4 + (x-x_4)\mathbf{A}_5 \\ & & & & \mathbf{P}(x) \end{bmatrix}. \tag{25b}$$

For general n, the block entries $\mathbf{U}_{j,n}$ are given as follows

$$\mathbf{U}_{j,n} = \mathbf{A}_j, \quad j = 1\colon n-2, \tag{25c}$$

$$\mathbf{U}_{n-1,n} = \mathbf{A}_{n-1} + (x - x_{n-1})\mathbf{A}_n, \tag{25d}$$

$$\mathbf{U}_{n,n} = \mathbf{P}(x). \tag{25e}$$

From (15) and (25a), it is clear that the conditioning of this case is highly dependent on the configuration of the nodes x_j. For reasonable configurations of the nodes (e.g. roots of unity or Chebyshev points on the real line), we get moderate condition numbers; otherwise, the problem is not usually well-conditioned. For more details, see [22]. If $x = x_j$ $(j = 0\colon n-2)$, different block pivoting strategies for different values of j can be considered. For more details, see [2].

3.4. Pochhammer basis

As a special case of the Newton basis, we consider the Pochhammer basis $\{(x+a)^{\overline{j}}\}_{j=0}^{n}$, where the Pochhammer symbol $(x+a)^{\overline{j}}$ denotes the rising factorial power, i.e.,

$$(x+a)^{\overline{j}} = \begin{cases} 1, & j = 0, \\ (x+a)(x+a+1)\cdots(x+a+j-1), & j > 0. \end{cases} \tag{26}$$

The Pochhammer symbol $(x + a)^{\overline{k}}$ arises in combinatorial applications and in the solution of difference equations [14]. The Pochhammer basis polynomials defined in (26) are an instance of Newton basis polynomials with nodes $x_j = -(a + j)$, $(j = 0\colon n-1)$. If we let $\alpha_j = 1$, $\beta_j = -(a + j)$ and $\gamma_j = 0$ in (6), then we will get the Pochhammer basis.

An $s \times s$ matrix polynomial $\mathbf{P}(x)$ written in the Pochhammer basis is

$$\mathbf{P}(x) = \sum_{j=0}^{n}(x+a)^{\overline{j}}\mathbf{A}_k. \tag{27}$$

The companion matrix pencil and **LU** factors can be found using exactly the same procedure described in Section 3.3.

4. Bernstein basis

A Bernstein polynomial (also called Bézier polynomial) defined over the interval $[a, b]$ has the form

$$b_j(x) = \binom{n}{j}\frac{(x-a)^j(b-x)^{n-j}}{(b-a)^n}, \quad j = 0\colon n. \tag{28}$$

This is not a typical scaling of the Bernstein polynomials; however, this scaling makes the companion pencils derived slightly easier to write. The Bernstein polynomials are nonnegative in $[a, b]$, i.e., $b_j(x) \geq 0$ for all $x \in [a, b]$ ($j = 0 : n$). Bernstein polynomials are widely used in computer-aided geometric design (e.g. see [11]).

An $s \times s$ polynomial matrix $\mathbf{P}(x)$ written in the Bernstein basis is of the form

$$\mathbf{P}(x) = \sum_{j=0}^{n} b_j(x)\,\mathbf{A}_j, \tag{29}$$

where $\mathbf{A}_j$ are sometimes called the Bézier coefficients ($j = 0 : n$).

4.1. Companion matrix pencil

A companion matrix pencil determined by (29) for $n = 5$ (with clear generalizations for all positive n) is (see [18])

$$\mathbf{C}_0 = \begin{bmatrix} \frac{5a}{b-a}\mathbf{I}_s & & & & -\frac{b}{b-a}\mathbf{A}_0 \\ \frac{b}{b-a}\mathbf{I}_s & \frac{4a}{2(b-a)}\mathbf{I}_s & & & -\frac{b}{b-a}\mathbf{A}_1 \\ & \frac{b}{b-a}\mathbf{I}_s & \frac{3a}{3(b-a)}\mathbf{I}_s & & -\frac{b}{b-a}\mathbf{A}_2 \\ & & \frac{b}{b-a}\mathbf{I}_s & \frac{2a}{4(b-a)}\mathbf{I}_s & -\frac{b}{b-a}\mathbf{A}_3 \\ & & & \frac{b}{b-a}\mathbf{I}_s & \frac{a}{5(b-a)}\mathbf{A}_5 - \frac{b}{b-a}\mathbf{A}_4 \end{bmatrix}, \tag{30}$$

$$\mathbf{C}_1 = \begin{bmatrix} \frac{5}{b-a}\mathbf{I}_s & & & & -\frac{1}{b-a}\mathbf{A}_0 \\ \frac{1}{b-a}\mathbf{I}_s & \frac{4}{2(b-a)}\mathbf{I}_s & & & -\frac{1}{b-a}\mathbf{A}_1 \\ & \frac{1}{b-a}\mathbf{I}_s & \frac{3}{3(b-a)}\mathbf{I}_s & & -\frac{1}{b-a}\mathbf{A}_2 \\ & & \frac{1}{b-a}\mathbf{I}_s & \frac{2}{4(b-a)}\mathbf{I}_s & -\frac{1}{b-a}\mathbf{A}_3 \\ & & & \frac{1}{b-a}\mathbf{I}_s & \frac{1}{5(b-a)}\mathbf{A}_5 - \frac{1}{b-a}\mathbf{A}_4 \end{bmatrix}. \tag{31}$$

4.2. Explicit LU factors and complexity

Let $\mathbf{P}(x)$ be expressed relative to a basis of Bernstein polynomials as in (29). Using (30) and (31), the matrix $x\mathbf{C}_1 - \mathbf{C}_0$ can be written as

$$(x\mathbf{C}_1 - \mathbf{C}_0) = \begin{bmatrix} \frac{5(x-a)}{b-a}\mathbf{I}_s & & & & -\frac{(x-b)}{b-a}\mathbf{A}_0 \\ \frac{(x-b)}{b-a}\mathbf{I}_s & \frac{4(x-a)}{2(b-a)}\mathbf{I}_s & & & -\frac{(x-b)}{b-a}\mathbf{A}_1 \\ & \frac{(x-b)}{b-a}\mathbf{I}_s & \frac{3(x-a)}{3(b-a)}\mathbf{I}_s & & -\frac{(x-b)}{b-a}\mathbf{A}_2 \\ & & \frac{(x-b)}{b-a}\mathbf{I}_s & \frac{2(x-a)}{4(b-a)}\mathbf{I}_s & -\frac{(x-b)}{b-a}\mathbf{A}_3 \\ & & & \frac{(x-b)}{b-a}\mathbf{I}_s & \frac{(x-a)}{5(b-a)}\mathbf{A}_5 - \frac{(x-b)}{b-a}\mathbf{A}_4 \end{bmatrix}. \tag{32}$$

Comparing (32) with (10), we can verify that, although the Bernstein basis is not a degree-graded basis, it is interestingly similar to a degree-graded basis. In fact it turns out that $\alpha_j(x) = \alpha(x) = \frac{b-x}{b-a}$, $\gamma_j(x) = 0$ and $k_{n-1}(x) = \frac{b-x}{b-a}$. Here, as opposed to what is described in Section 2, α and k_{n-1} are not constant values and (6) is not valid anymore. Having this information is enough for us to be able to find the **LU** factors of a matrix of the form (32) and of degree n using (11). If $x \neq a$, $x\mathbf{C}_1 - \mathbf{C}_0 = \mathbf{LU}$ where

$$\mathbf{L} = \begin{bmatrix} \mathbf{I}_s & & & & \\ -\frac{b-x}{5(x-a)}\mathbf{I}_s & \mathbf{I}_s & & & \\ & -\frac{2(b-x)}{4(x-a)}\mathbf{I}_s & \mathbf{I}_s & & \\ & & -\frac{3(b-x)}{3(x-a)}\mathbf{I}_s & \mathbf{I}_s & \\ & & & -\frac{4(b-x)}{2(x-a)}\mathbf{I}_s & \mathbf{I}_s \end{bmatrix} \tag{33a}$$

$$\mathbf{U} = \begin{bmatrix} \frac{5(x-a)}{b-a}\mathbf{I}_s & & & & \mathbf{U}_{1,5} \\ & \frac{4(x-a)}{2(b-a)}\mathbf{I}_s & & & \mathbf{U}_{2,5} \\ & & \frac{3(x-a)}{3(b-a)}\mathbf{I}_s & & \mathbf{U}_{3,5} \\ & & & \frac{2(x-a)}{4(b-a)}\mathbf{I}_s & \mathbf{U}_{4,5} \\ & & & & \mathbf{U}_{5,5} \end{bmatrix}, \tag{33b}$$

where for general n, the block entries $\mathbf{U}_{j,n}$ are given as follows

$$\mathbf{U}_{1,n} = \frac{b-x}{b-a}\mathbf{A}_0, \tag{33c}$$

$$\mathbf{U}_{j,n} = \frac{b-x}{b-a}\mathbf{A}_{j-1} + \frac{(j-1)(b-x)}{(n-j+2)(x-a)}\mathbf{U}_{j-1,n}, \quad j = 2:n-1, \tag{33d}$$

$$\mathbf{U}_{n,n} = \frac{(b-a)^{n-1}}{n(x-a)^{n-1}}\mathbf{P}(x). \tag{33e}$$

As with the preceding cases, the factors $\mathbf{L}$ and $\mathbf{U}$ in (33) can be assembled in $\mathcal{O}(ns^2)$ work and the linear systems (3) using these factors can be solved in $\mathcal{O}(ns^2 + s^3)$ work.

4.3. Block pivoting strategies for stabilization of LU factors

Consider the generalized companion matrix pencil of the form (32) for a matrix polynomial $\mathbf{P}(x)$ written using the Bernstein basis. Due to the similarities between the Bernstein basis and the degree-graded bases (described in Section 4.2), we can use the same matrix $\mathbf{S}$ in (12) for block pivoting of a matrix of the form (31) and of degree n when $x \to a$. In that case, similar to (14) we will have $\mathbf{S}(x\mathbf{C}_1 - \mathbf{C}_0) = \mathbf{LU}$ where:

$$\mathbf{L} = \begin{bmatrix} \mathbf{I}_s & & & & \\ & \mathbf{I}_s & & & \\ & & \mathbf{I}_s & & \\ & & & \mathbf{I}_s & \\ -\frac{\binom{5}{1}(x-a)}{(b-x)}\mathbf{I}_s & -\frac{\binom{5}{2}(x-a)^2}{(b-x)^2}\mathbf{I}_s & -\frac{\binom{5}{3}(x-a)^3}{(b-x)^3}\mathbf{I}_s & -\frac{\binom{5}{4}(x-a)^4}{(b-x)^4}\mathbf{I}_s & \mathbf{I}_s \end{bmatrix}, \tag{34a}$$

$$\mathbf{U} = \begin{bmatrix} -\frac{b-x}{b-a}\mathbf{I}_s & \frac{4(x-a)}{2(b-a)}\mathbf{I}_s & & & \frac{b-x}{b-a}\mathbf{A}_1 \\ & -\frac{b-x}{b-a}\mathbf{I}_s & \frac{3(x-a)}{3(b-a)}\mathbf{I}_s & & \frac{b-x}{b-a}\mathbf{A}_2 \\ & & -\frac{b-x}{b-a}\mathbf{I}_s & \frac{2(x-a)}{4(b-a)}\mathbf{I}_s & \frac{b-x}{b-a}\mathbf{A}_3 \\ & & & -\frac{b-x}{b-a}\mathbf{I}_s & \frac{b-x}{b-a}\mathbf{A}_4 + \frac{(x-a)}{5(b-a)}\mathbf{A}_5 \\ & & & & \frac{(b-a)^4}{(b-x)^4}\mathbf{P}(x) \end{bmatrix}. \tag{34b}$$

For general n, the block entries $\mathbf{U}_{j,n}$ are given as follows

$$\mathbf{U}_{j,n} = \frac{b-x}{b-a}\mathbf{A}_j, \quad j = 1:n-2, \tag{34c}$$

$$\mathbf{U}_{n-1,n} = \frac{b-x}{b-a}\mathbf{A}_{n-1} + \frac{(x-a)}{n(b-a)}\mathbf{A}_n, \tag{34d}$$

$$\mathbf{U}_{n,n} = \frac{(b-a)^{n-1}}{(b-x)^{n-1}}\mathbf{P}(x). \tag{34e}$$

The cost of building (34) and the cost of solving the system (3) by that do not change comparing to the costs of (33). However, it should be noted that the factors in (34) are not appropriate for x values such that $|\frac{x-a}{b-x}| \gg \frac{1}{n}$. For more details, see Appendix.

5. Lagrange basis

An $s \times s$ matrix polynomial of degree n may be specified by data $\{(x_j, \mathbf{P}_j)\}_{j=0}^{n}$ instead of the monomial basis coefficients $\mathbf{A}_j$ $(j = 0{:}n)$ as in Section 3.2. We assume that the nodes x_j are distinct sample points. In this case, it is natural to express $\mathbf{P}(x)$ using the Lagrange basis. The Lagrange basis consists of the polynomials $\{\ell_j(x)\}_{j=0}^{n}$, where

$$\ell_j(x) := w_j \prod_{\substack{k=0\\k\neq j}}^{n} (x - x_k), \quad j = 0{:}n, \tag{35a}$$

$$w_j := \prod_{\substack{k=0\\k\neq j}}^{n} \frac{1}{(x_j - x_k)}, \quad j = 0{:}n. \tag{35b}$$

The scalar factors w_j $(j = 0{:}n)$ in (35b) are the barycentric weights. Then, the matrix polynomial $\mathbf{P}(x)$ expressed in the Lagrange basis $\{\ell_j(x)\}_{j=0}^{n}$ is

$$\mathbf{P}(x) = \sum_{j=0}^{n} \ell_j(x)\, \mathbf{P}_j. \tag{36}$$

The matrix polynomial $\mathbf{P}(x)$ can be expressed equivalently in barycentric form (e.g., see [6,16])

$$\mathbf{P}(x) = \ell(x) \sum_{j=0}^{n} \frac{w_j}{x - x_j} \mathbf{P}_j \quad \text{where} \tag{37a}$$

$$\ell(x) := \prod_{j=0}^{n} (x - x_j) = (x - x_0)(x - x_1) \cdots (x - x_n). \tag{37b}$$

Lagrange polynomial interpolation is traditionally viewed strictly as a tool for theoretical analysis; however, recent work reveals several advantages to computation using interpolating polynomials in the barycentric Lagrange form.

5.1. Companion matrix pencil

A companion matrix pencil $(\mathbf{C}_0, \mathbf{C}_1)$ associated with the polynomial $\mathbf{P}(x)$ expressed in Lagrange form is given by

$$\mathbf{C}_0 = \begin{bmatrix} x_0\mathbf{I}_s & & & \mathbf{P}_0 \\ & \ddots & & \vdots \\ & & x_n\mathbf{I}_s & \mathbf{P}_n \\ -w_0\mathbf{I}_s & \cdots & -w_n\mathbf{I}_s & \mathbf{0}_s \end{bmatrix}, \tag{38a}$$

$$\mathbf{C}_1 = \mathbf{diag}\,(\mathbf{I}_n \otimes \mathbf{I}_s, \mathbf{0}_s)\,. \tag{38b}$$

The matrix pencil (38) stands out from all the other pencils we derive in that the matrices are of dimension $(n+2)s \times (n+2)s$ rather than $ns \times ns$. As such, if $\mathbf{P}(x)$ is nonsingular, the pencil admits $2s$ eigenvalues at infinity in addition to the polynomial eigenvalues of $\mathbf{P}(x)$. For more details, see [3,4].

5.2. Explicit LU factors and complexity

The matrices $\mathbf{C}_0$ and $\mathbf{C}_1$ in (38) for $n = 3$ and clear generalizations for all positive n imply that

$$x\mathbf{C}_1 - \mathbf{C}_0 = \begin{bmatrix} (x-x_0)\mathbf{I}_s & & & & -\mathbf{P}_0 \\ & (x-x_1)\mathbf{I}_s & & & -\mathbf{P}_1 \\ & & (x-x_2)\mathbf{I}_s & & -\mathbf{P}_2 \\ & & & (x-x_3)\mathbf{I}_s & -\mathbf{P}_2 \\ w_0\mathbf{I}_s & w_1\mathbf{I}_s & w_2\mathbf{I}_s & w_3\mathbf{I}_s & \mathbf{0}_s \end{bmatrix}. \tag{39}$$

If x coincides with any of the nodes x_j, then the $(j+1, j+1)$ block in (39) is zero, requiring block pivoting in the construction of the **LU** factors. Assuming that $x \neq x_j$ for $j = 0{:}n$, the matrix in (39) admits a block **LU** factoring $x\mathbf{C}_1 - \mathbf{C}_0 = \mathbf{LU}$, where

$$\mathbf{L} = \begin{bmatrix} \mathbf{I}_s & & & & \\ & \mathbf{I}_s & & & \\ & & \mathbf{I}_s & & \\ & & & \mathbf{I}_s & \\ \frac{w_0}{x-x_0}\mathbf{I}_s & \frac{w_1}{x-x_1}\mathbf{I}_s & \frac{w_2}{x-x_2}\mathbf{I}_s & \frac{w_3}{x-x_3}\mathbf{I}_s & \mathbf{I}_s \end{bmatrix}, \tag{40a}$$

$$\mathbf{U} = \begin{bmatrix} (x-x_0)\mathbf{I}_s & & & & -\mathbf{P}_0 \\ & (x-x_1)\mathbf{I}_s & & & -\mathbf{P}_1 \\ & & (x-x_2)\mathbf{I}_s & & -\mathbf{P}_2 \\ & & & (x-x_3)\mathbf{I}_s & -\mathbf{P}_3 \\ & & & & \frac{1}{\ell(x)}\mathbf{P}(x) \end{bmatrix}. \tag{40b}$$

The block matrices in (40) imply that $x\mathbf{C}_1 - \mathbf{C}_0$ is singular iff the matrix $\mathbf{P}(x)$ is singular. The weights w_j can be precomputed in $\mathcal{O}(n^2)$ operations ($j = 0{:}n$). Assembling the block matrices **L** and **U** requires $\mathcal{O}(ns^2)$ operations with most of the work in using (37a) to compute the bottom right block $\mathbf{U}_{n+2,n+2}$, i.e.,

$$\mathbf{U}_{n+2,n+2} = \frac{1}{\ell(x)}\mathbf{P}(x) = \sum_{j=0}^{n} \frac{w_j}{x-x_j}\mathbf{P}_j. \tag{41}$$

Moreover, the factoring in (40) implies that, the cost of solving a system such as (3) using (39) is $\mathcal{O}(ns^2 + s^3)$.

5.3. Block pivoting strategies for stabilization of LU factors

Just as for the Newton basis (Section 3.3), the conditioning of the problems such as (40) in the Lagrange basis is highly dependent on the configuration of the nodes x_j in general.

The matrix (39) factors directly as in (40) when x does not coincide with any of the interpolation nodes (which are also the zeros of the Lagrange polynomials (35)). Considering first the node x_0, as $x \to x_0$, it is preferable to premultiply (39) by the block permutation matrix $\mathbf{S}(x_0) := \mathbf{S}$ as in (12). Then,

$$\mathbf{S}(x_0)\,(x_0\,\mathbf{C}_1 - \mathbf{C}_0) = \mathbf{L}(x_0)\mathbf{U}(x_0) \tag{42}$$

where

$$\mathbf{L}(x_0) = \mathbf{I}_{n+2} \otimes \mathbf{I}_s, \tag{43a}$$

$$\mathbf{U}(x_0) = \begin{bmatrix} w_0\mathbf{I}_s & w_1\mathbf{I}_s & \cdots & w_n\mathbf{I}_s & \mathbf{0}_s \\ & (x_0-x_1)\mathbf{I}_s & & & -\mathbf{P}_1 \\ & & \ddots & & \vdots \\ & & & (x_0-x_n)\mathbf{I}_s & -\mathbf{P}_n \\ & & & & -\mathbf{P}_0 \end{bmatrix}. \tag{43b}$$

More generally, if x is close to one of the other interpolation nodes x_j (i.e., $|x - x_j| < \epsilon$ for some small ϵ ($j = 1:n$)), we find

$$\mathbf{S}(x_j)\left(x_j\mathbf{C}_1 - \mathbf{C}_0\right) = \mathbf{L}(x_j)\mathbf{U}(x_j),$$

with

$$\mathbf{S}(x_j) = \left(\mathbf{E}_{[n\leftrightarrow n+1]}\mathbf{E}_{[n-1\leftrightarrow n]}\cdots\mathbf{E}_{[j+1\leftrightarrow j+2]}\right)\otimes\mathbf{I}_s, \tag{44a}$$

$$\mathbf{L}(x_j) = (\mathbf{I}_n + \mathbf{e}_{j+1}\boldsymbol{m}(x_j)^{\mathrm{T}})\otimes\mathbf{I}_s, \tag{44b}$$

$$m_k(x_j) = \begin{cases} w_{k-1}/(x_j - x_{k-1}), & k = 1:j,\\ 0, & k = j+1:n, \end{cases} \tag{44c}$$

$$\mathbf{U}(x_j) = \begin{bmatrix} (x_j - x_0)\mathbf{I}_s & & & & & & & -\mathbf{P}_0\\ & \ddots & & & & & & \vdots\\ & & (x_j - x_{j-1})\mathbf{I}_s & & & & & -\mathbf{P}_{j-1}\\ & & & w_j\mathbf{I}_s & w_{j+1}\mathbf{I}_s & \cdots & w_n\mathbf{I}_s & \sum_{i=0}^{j-1}\frac{w_i}{x_j - x_i}\mathbf{P}_i\\ & & & & (x_j - x_{j+1})\mathbf{I}_s & & & -\mathbf{P}_{j+1}\\ & & & & & \ddots & & \vdots\\ & & & & & & (x_j - x_n)\mathbf{I}_s & -\mathbf{P}_n\\ & & & & & & & -\mathbf{P}_j \end{bmatrix}. \tag{44d}$$

6. Concluding remarks

We have collected and exhibited a number of useful block **LU** factorings of generalized companion matrix pencils associated with matrix polynomials expressed in polynomial bases other than the standard monomial basis. These generalized companion matrix pencils and their **LU** factors allow work to be carried out directly in terms of the basis prescribed, avoiding any potential ill-conditioned change of basis (particularly in changing from the Lagrange basis to the monomial basis). Moreover, we have given necessary prescriptions for pivoting and alternative factorings for use when an approximate polynomial eigenvalue x happens to be a zero of a basis function—i.e., $x = 0$ in the monomial case, $x = x_k$ is one of the nodes in Lagrange or Newton case, $x = a$ in the Bernstein case, or x is a zero of any of $\phi_0(x), \ldots, \phi_n(x)$ in the orthogonal polynomial case.

In future work, we will discuss rules and heuristics for choosing the tolerance ϵ that should be used to determine when the value x is in a neighbourhood of such a point (except in the case of Bernstein polynomials). It seems clear that more experiments are necessary to establish guidelines, particularly for the Lagrange case, where the interplay between the geometry of the nodes and the accuracy of the eigenvalues plays a very strong role.

Again, in future work, we hope to establish that if x is very close to a simple polynomial eigenvalue of $\mathbf{P}(x)$, then the singularity of the pencil $x\mathbf{C}_1 - \mathbf{C}_0$ is harmless as it is in the usual linear eigenvalue case using inverse iteration. Our experiments support that this is in fact the case and our expectation is that the proof for the linear case should extend readily to the case of polynomial eigenvalues. This conjecture, that the near-singular behaviour of $\mathbf{P}(x)$ is harmless for eigenvector computation, is believable but still requires proof.

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Appendix

We want to show that the factors in (34) are not appropriate for x values such that $|\frac{x-a}{b-x}| \gg \frac{1}{n}$. From (34a), we have

$$\kappa_{\infty}(\mathbf{L}) = \max\left(1, \left(\sum_{k=1}^{n-1}\binom{n}{k}\left|\frac{x-a}{x-b}\right|^{k}\right)^{2}\right) = \max\left(1, \left(\left(1+\left|\frac{x-a}{x-b}\right|\right)^{n} - 1 - \left|\frac{x-a}{x-b}\right|^{n}\right)^{2}\right), \tag{45}$$

where $\kappa_{\infty}(\mathbf{L})$ is the ∞-norm condition number of $\mathbf{L}$.

Now for some $h \in \mathbb{R}^{+}$, let $|\frac{x-a}{b-x}| = \frac{h}{n}$, then

$$\lim_{n\longrightarrow\infty}\left(\left(1+\left|\frac{x-a}{x-b}\right|\right)^{n} - 1 - \left|\frac{x-a}{x-b}\right|^{n}\right)^{2} = (\exp(h) - 1)^{2}. \tag{46}$$

Therefore, when $h \gg 1$ the condition number becomes large.

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